

THE S-BASIS AND M-BASIS PROBLEMS FOR SEPARABLE BANACH SPACES

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ABSTRACT. This note has two objectives. The first objective is show that, even if a separable Banach space does not have a Schauder basis (S-basis), there always exists Hilbert spaces \mathcal{H}_1 and \mathcal{H}_2 , such that \mathcal{H}_1 is a continuous dense embedding in \mathcal{B} and \mathcal{B} is a continuous dense embedding in \mathcal{H}_2 . This is the best possible improvement of a theorem due to Mazur (see [BA] and also [PE1]). The second objective is show how \mathcal{H}_2 allows us to provide a positive answer to the Marcinkiewicz-basis (M-basis) problem.

1. INTRODUCTION

Definition 1.1. Let \mathcal{B} separable Banach space, with dual space \mathcal{B}^* . A sequence $(x_n) \in \mathcal{B}$ is called a *S-basis* for \mathcal{B} if $\|x_n\|_{\mathcal{B}} = 1$ and, for each $x \in \mathcal{B}$, there is a unique sequence (a_n) of scalars such that

$$x = \lim_{n \rightarrow \infty} \sum_{k=1}^n a_k x_k = \sum_{k=1}^{\infty} a_k x_k.$$

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Definition 1.2. Let $\langle \{x_i : i \in \mathbb{N}\} \rangle$ be the set of all linear combinations of the family of vectors $\{x_i\}$ (linear span). The family $\{(x_i, x_i^*)\}_{i=1}^\infty \subset \mathcal{B} \times \mathcal{B}^*$ is called:

- (1) A fundamental system if $\overline{\langle \{x_i : i \in \mathbb{N}\} \rangle} = \mathcal{B}$.
- (2) A minimal system if $x_j \notin \overline{\langle \{x_i : i \in \mathbb{N} \setminus \{j\}\} \rangle}$.
- (3) A total if for each $x \neq 0$ there exists $i \in \mathbb{N}$ such that $x_i^*(x) \neq 0$.
- (4) A biorthogonal system if $x_i^*(x_j) = \delta_{ij}$, for all $i, j \in \mathbb{N}$.
- (5) A M-basis if it is a fundamental minimal, total and biorthogonal system.

The first problem we consider had its beginning with a question raised by Banach. He asked whether every separable Banach space has a S-basis. Mazur gave a partial answer. He proved that every infinite-dimensional separable Banach space contains an infinite-dimensional subspace with a S-basis.

In 1972, Enflo [EN] answered Banach's question in the negative by providing a separable Banach space \mathcal{B} , without a S-basis and without the approximation property (i.e., every compact operator on \mathcal{B} is the limit of a sequence of finite rank operators). Every Banach space with a S-basis has the approximation property and Grothendieck [GR] proved that if a Banach space had the approximation property, then it would also have a S-basis. In the first section we show that, given \mathcal{B} there exists separable Hilbert spaces \mathcal{H}_1 and \mathcal{H}_2 such that $\mathcal{H}_1 \subset \mathcal{B} \subset \mathcal{H}_2$ as continuous dense embeddings. The

existence of \mathcal{H}_1 is the best possible improvement of Mazur's Theorem, while the existence of \mathcal{H}_2 shows that \mathcal{B} is very close to the best possible case in a well-defined manner.

The second problem we consider is associated with a weaker structure discovered by Marcinkiewicz [M]. He showed that every separable Banach space \mathcal{B} has a biorthogonal system $\{x_n, x_n^*\}$, with $\overline{\langle \{x_n\} \rangle} = \mathcal{B}$. This system has many of the properties of an S-basis and is now known as a M-basis for \mathcal{B} . A well-known open problem for the M-basis is whether one can choose the system $\{x_n, x_n^*\}$ such that $\|x_n\| \|x_n^*\| = 1$ (see Diestel [D]). This is called the M-basis problem for separable Banach spaces. It has been studied by Singer [SI], Davis and Johnson [DJ], Ovsepian and Pelczyński [OP], Pelczyński [PE] and Plichko [PL]. The work of Ovsepian and Pelczyński [OP] led to the construction of a bounded M-basis, while that of Pelczyński [PE] and Plichko [PL] led to independent proofs that, for every $\varepsilon > 0$, it is possible to find a biorthogonal system with the property that $\|x_n\| \|x_n^*\| < 1 + \varepsilon$. The question of whether we can set $\varepsilon = 0$ has remained unanswered since 1976. In this case, we provide a positive answer by constructing a biorthogonal system with the property that $\|x_n\| \|x_n^*\| = 1$.

2. THE S-BASIS PROBLEM

In this section, we construct our Hilbert space rigging of any given separable Banach space as continuous dense embeddings. We begin with the construction of \mathcal{H}_2 .

Theorem 2.1. *Suppose \mathcal{B} is a separable Banach space, then there exist a separable Hilbert space \mathcal{H}_2 such that, $\mathcal{B} \subset \mathcal{H}_2$ as a continuous dense embedding.*

Proof. Let $\{x_n\}$ be a countable dense sequence in \mathcal{B} and let $\{x_n^*\}$ be any fixed set of corresponding duality mappings (i.e., $x_n^* \in \mathcal{B}^*$, the dual space of \mathcal{B} and $x_n^*(x_n) = \langle x_n, x_n^* \rangle = \|x_n\|_{\mathcal{B}}^2 = \|x_n^*\|_{\mathcal{B}^*}^2$). For each n , let $t_n = \frac{1}{\|x_n^*\|^2 2^n}$ and define (u, v) by:

$$(u, v) = \sum_{n=1}^{\infty} t_n x_n^*(u) \bar{x}_n^*(v) = \sum_{n=1}^{\infty} \frac{1}{\|x_n^*\|^2 2^n} x_n^*(u) \bar{x}_n^*(v).$$

It is easy to see that (u, v) is an inner product on \mathcal{B} . Let \mathcal{H}_2 be the completion of \mathcal{B} with respect to this inner product. It is clear that \mathcal{B} is dense in \mathcal{H}_2 , and

$$\|u\|_{\mathcal{H}_2}^2 = \sum_{n=1}^{\infty} t_n |x_n^*(u)|^2 \leq \sup_n \frac{1}{\|x_n^*\|^2} |x_n^*(u)|^2 = \|u\|_{\mathcal{B}}^2,$$

so the embedding is continuous. \square

In order to construct our second Hilbert space, we need the following result by Lax [L].

Theorem 2.2 (Lax). *Let $A \in L[\mathcal{B}]$. If A is selfadjoint on \mathcal{H}_2 (i.e., $(Ax, y)_{\mathcal{H}_2} = (x, Ay)_{\mathcal{H}_2}, \forall x, y \in \mathcal{B}$), then A has a bounded extension to \mathcal{H}_2 and $\|A\|_{\mathcal{H}_2} \leq M \|A\|_{\mathcal{B}}$ for some positive constant M .*

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Proof. Let $x \in \mathcal{B}$ and, without loss, we can assume that $M = 1$ and $\|x\|_{\mathcal{H}_2} =$

1. Since A is selfadjoint,

$$\|Ax\|_{\mathcal{H}_2}^2 = (Ax, Ax) = (x, A^2x) \leq \|x\|_{\mathcal{H}_2} \|A^2x\|_{\mathcal{H}_2} = \|A^2x\|_{\mathcal{H}_2}.$$

Thus, we have $\|Ax\|_{\mathcal{H}_2}^4 \leq \|A^4x\|_{\mathcal{H}_2}$, so it is easy to see that $\|Ax\|_{\mathcal{H}_2}^{2n} \leq \|A^{2n}x\|_{\mathcal{H}_2}$ for all n . It follows that:

$$\begin{aligned} \|Ax\|_{\mathcal{H}_2} &\leq (\|A^{2n}x\|_{\mathcal{H}_2})^{1/2n} \leq (\|A^{2n}x\|_{\mathcal{B}})^{1/2n} \\ &\leq (\|A^{2n}\|_{\mathcal{B}})^{1/2n} (\|x\|_{\mathcal{B}})^{1/2n} \leq \|A\|_{\mathcal{B}} (\|x\|_{\mathcal{B}})^{1/2n}. \end{aligned}$$

Letting $n \rightarrow \infty$, we get that $\|Ax\|_{\mathcal{H}_2} \leq \|A\|_{\mathcal{B}}$ for any x in the dense set of the unit ball $B_{\mathcal{H}_2} \cap \mathcal{B}$. Since the norm is attained on a dense set of the unit ball, we are done. \square

For our second Hilbert space, fix \mathcal{B} and define \mathcal{H}_1 by:

$$\begin{aligned} \mathcal{H}_1 &= \left\{ u \in \mathcal{B} \mid \sum_{n=1}^{\infty} t_n^{-1} |(u, x_n)_2|^2 < \infty \right\}, \quad \text{with} \\ (u, v)_1 &= \sum_{n=1}^{\infty} t_n^{-1} (u, x_n)_2 (x_n, v)_2. \end{aligned}$$

For $u \in \mathcal{B}$, let $T_{12}u$ be defined by $T_{12}u = \sum_{n=1}^{\infty} t_n (u, x_n)_2 x_n$.

Theorem 2.3. *The operator T_{12} is a positive trace class operator on \mathcal{B} with a bounded extension to \mathcal{H}_2 . In addition, $\mathcal{H}_1 \subset \mathcal{B} \subset \mathcal{H}_2$ (as continuous dense embeddings), $(T_{12}^{1/2}u, T_{12}^{1/2}v)_1 = (u, v)_2$ and $(T_{12}^{-1/2}u, T_{12}^{-1/2}v)_2 = (u, v)_1$.*

Proof. First, since terms of the form $\{u_N = \sum_{k=1}^N t_k^{-1} (u, x_k)_2 x_k : u \in \mathcal{B}\}$ are dense in \mathcal{B} , we see that \mathcal{H}_1 is dense in \mathcal{B} . It follows that \mathcal{H}_1 is also dense in \mathcal{H}_2 .

For the operator T_{12} , we see that $\mathcal{B} \subset \mathcal{H}_2 \Rightarrow (u, x_n)_2$ is defined for all $u \in \mathcal{B}$, so that T_{12} maps $\mathcal{B} \rightarrow \mathcal{B}$ and:

$$\|T_{12}u\|_{\mathcal{B}}^2 \leq \left[\sum_{n=1}^{\infty} t_n^2 \|x_n\|_{\mathcal{B}}^2 \right] \left[\sum_{n=1}^{\infty} |(u, x_n)_2|^2 \right] = M \|u\|_2^2 \leq M \|u\|_{\mathcal{B}}^2.$$

Thus, T_{12} is a bounded operator on \mathcal{B} . It is clearly trace class and, since $(T_{12}u, u)_2 = \sum_{n=1}^{\infty} t_n |(u, x_n)_2|^2 > 0$, it is positive. From here, it's easy to see that T_{12} is selfadjoint on \mathcal{H}_2 so, by Theorem 2.2 it has a bounded extension to \mathcal{H}_2 .

An easy calculation now shows that $(T_{12}^{1/2}u, T_{12}^{1/2}v)_1 = (u, v)_2$ and $(T_{12}^{-1/2}u, T_{12}^{-1/2}v)_2 = (u, v)_1$. \square

Since the counter example of Enflo, the only direct information about a Banach space without a basis has been the following theorem of Mazur:

Theorem 2.4. *Every infinite dimensional separable Banach contains a infinite dimensional subspace with a basis.*

Theorems 2.1 and 2.3 show that, even if a Banach space does not have a basis, it is very close to the best possible case.

Remark 2.5. *Historically, Gross [G] first proved that every real separable Banach space \mathcal{B} contains a separable Hilbert space (version of \mathcal{H}_1), as a dense embedding, and that this space is the support of a Gaussian measure. Then Kuelbs [KB] showed that one can construct \mathcal{H}_2 so that $\mathcal{H}_1 \subset \mathcal{B} \subset \mathcal{H}_2$ as continuous dense embeddings, with \mathcal{H}_1 and \mathcal{H}_2 related by Theorem 2.3.*

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A particular Gross-Kuelbs construction of \mathcal{H}_2 was used in [GZ] to provide the foundations for the Feynman path integral formulation of quantum mechanics [FH] (see also [GZ1]).

This construction was also used in [GBZS] to show that every bounded linear operator A on a separable Banach space \mathcal{B} has a adjoint A^* defined on \mathcal{B} , such that (see below):

- (1) A^*A is m -accretive (i.e., if $x \in \mathcal{B}$ and x^* is a corresponding duality, then $\langle A^*Ax, x^* \rangle \geq 0$),
- (2) $(A^*A)^* = A^*A$ (selfadjoint), and
- (3) $I + A^*A$ has a bounded inverse.

Example 2.6. The following example shows how easy it is to construct an adjoint A^* satisfying all the above conditions, using only \mathcal{H}_2 . Let Ω be a bounded open domain in \mathbb{R}^n with a class \mathbb{C}^1 boundary and let $\mathcal{H}_0^1[\Omega]$, be the set of all real-valued functions $u \in L^2[\Omega]$ such that their first order weak partial derivatives are in $L^2[\Omega]$ and vanish on the boundary. It follows that

$$(u, v) = \int_{\Omega} \nabla u(\mathbf{x}) \cdot \nabla v(\mathbf{x}) d\mathbf{x} = \langle u, J_0 v \rangle,$$

defines an inner product on $\mathcal{H}_0^1[\Omega]$, where J_0 is the conjugate isomorphism between $\mathcal{H}_0^1[\Omega]$ and its dual $\mathcal{H}^{-1}[\Omega]$. The space $\mathcal{H}^{-1}[\Omega]$ coincides with the set of all distributions of the form

$$u = h_0 + \sum_{i=1}^n \frac{\partial h_i}{\partial x_i}, \quad \text{where } h_i \in L^2[\Omega], \quad 1 \leq i \leq n.$$

In this case we also have for $p \in [2, \infty)$ and $q \in (1, 2]$, $\frac{1}{p} + \frac{1}{q} = 1$ that,

$$\mathcal{H}_0^1[\Omega] \subset L^p[\Omega] \subset L^q[\Omega] \subset \mathcal{H}^{-1}[\Omega]$$

all as continuous dense embeddings.

From the inner product on $\mathcal{H}_0^1[\Omega]$ we see that $J_0 = -\Delta$, the Laplace operator under Dirichlet homogeneous boundary conditions on Ω . If we set $\mathcal{H}_1 = \mathcal{H}_0^1[\Omega]$, $\mathcal{H}_2 = \mathcal{H}^{-1}$ and $J = J_0^{-1}$, then for every $A \in \mathcal{C}[L^p(\Omega)]$ (i.e., the closed densely defined linear operators on $L^p(\Omega)$), we obtain $A^* \in \mathcal{C}[L^p(\Omega)]$, where $A^* = J^{-1}A'J|_p = [-\Delta]A'[-\Delta]^{-1}|_p$ for each $A' \in \mathcal{C}[L^q(\Omega)]$. It is now easy to show that A^* satisfies the conditions (1)-(3) above for an adjoint operator on $L^p(\Omega)$.

3. THE M-BASIS PROBLEM

To understand the M-basis problem and its solution in a well-known setting, let \mathbb{R}^2 have its standard inner product (\cdot, \cdot) and let x_1, x_2 be any two independent basis vectors. Define a new inner product on \mathbb{R}^2 by

$$\begin{aligned} \langle y | z \rangle &= t_1 (x_1 \otimes x_1) (y \otimes z) + t_2 (x_2 \otimes x_2) (y \otimes z) \\ (3.1) \quad &= t_1 (y, x_1) (z, x_1) + t_2 (y, x_2) (z, x_2), \end{aligned}$$

where $t_1, t_2 > 0$, $t_1 + t_2 = 1$. Define new functionals S_1 and S_2 by:

$$S_1(x) = \frac{\langle x | x_1 \rangle}{\alpha_1 \langle x_1 | x_1 \rangle}, \quad S_2(x) = \frac{\langle x | x_2 \rangle}{\alpha_2 \langle x_2 | x_2 \rangle}, \quad \text{for } y \in \mathbb{R}^2.$$

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Where $\alpha_1, \alpha_2 > 0$ are chosen to ensure that $\|S_1\| = \|S_2\| = 1$. Note that, if

$(x_1, x_2) = 0$, S_1 and S_2 reduce to

$$S_1(x) = \frac{(x, x_1)}{\alpha_1 \|x_1\|}, \quad S_2(x) = \frac{(x, x_2)}{\alpha_2 \|x_2\|}.$$

Thus, we can define many equivalent inner products on \mathbb{R}^2 and many linear functionals with the same properties but different norms.

The following example shows how this construction can be of use.

Example 3.1. *In this example, let $x_1 = e_1$ and $x_2 = e_1 + e_2$, where $e_1 = (1, 0)$, $e_2 = (0, 1)$. In this case, the biorthogonal functionals are generated by the vectors $\bar{x}_1 = e_1 - e_2$ and $\bar{x}_2 = e_2$ (i.e., $x_1^*(x) = (x, \bar{x}_1)$, $x_2^*(x) = (x, \bar{x}_2)$). It follows that $(x_1, \bar{x}_2) = 0$, $(x_1, \bar{x}_1) = 1$ and $(x_2, \bar{x}_1) = 0$, $(x_2, \bar{x}_2) = 1$. However, $\|x_1\| \|\bar{x}_1\| = \sqrt{2}$, $\|x_2\| \|\bar{x}_2\| = \sqrt{2}$, so that $\{x_1, (\cdot, \bar{x}_1)\}$ and $\{x_2, (\cdot, \bar{x}_2)\}$ fails to solve the M-basis problem on \mathbb{R}^2 .*

In this case, we set $\alpha_1 = 1$ and $\alpha_2 = \|x_2\|$ so that, without changing x_1 and x_2 , and using the inner product from equation (1.1) in the form

$$\langle x | y \rangle = t_1 (x, \bar{x}_1) (y, \bar{x}_1) + t_2 (x, \bar{x}_2) (y, \bar{x}_2),$$

S_1 and S_2 become

$$S_1(x) = \frac{(x, \bar{x}_1)}{\|\bar{x}_1\|}, \quad S_2(x) = \frac{(x, \bar{x}_2)}{\|x_2\|}.$$

It now follows that $S_i(x_i) = 1$ and $S_i(x_j) = 0$ for $i \neq j$ and $\|S_i\| \|x_i\| = 1$, so that system $\{x_1, S_1\}$ and $\{x_2, S_2\}$ solves the M-basis problem.

Remark 3.2. *For a given set of independent vectors on a finite dimensional vector space, It is known that the corresponding biorthogonal functionals are unique. This example shows that uniqueness is only up to a scale factor and this is what we need to produce an M-basis.*

The following theorem shows how our solution to the M-basis problem for \mathbb{R}^2 can be extended to any separable Banach space.

Theorem 3.3. *Let \mathcal{B} be a infinite-dimensional separable Banach space. Then \mathcal{B} contains an M-basis with the property that $\|x_i\|_{\mathcal{B}} \|x_i^*\|_{\mathcal{B}^*} = 1$ for all i .*

Proof. Construct $\mathcal{H} = \mathcal{H}_2$ via Theorem 2.1, so that $\mathcal{B} \subset \mathcal{H}$ is a dense continuous embedding and let $\{x_i\}_{i=1}^{\infty}$ be a fundamental minimal system for \mathcal{B} . If $i \in \mathbb{N}$, let $M_{i,\mathcal{H}}$ be the closure of the span of $\{x_i\}$ in \mathcal{H} . Thus, $x_i \notin M_{i,\mathcal{H}}^{\perp}$, $M_{i,\mathcal{H}} \oplus M_{i,\mathcal{H}}^{\perp} = \mathcal{H}$ and $(y, x_i)_{\mathcal{H}} = 0$ for all $y \in M_{i,\mathcal{H}}^{\perp}$.

Let \hat{M}_i be the closure of the span of $\{x_j \mid j \neq i\}$ in \mathcal{B} . Since $\hat{M}_i \subset M_{i,\mathcal{H}}^{\perp}$ and $x_i \notin \hat{M}_i$, $(y, x_i)_{\mathcal{H}} = 0$ for all $y \in \hat{M}_i$. Let the seminorm $p_i(\cdot)$ be defined on the closure of the span of $\{x_i\}$, in \mathcal{B} by $p_i(y) = \|x_i\|_{\mathcal{B}} \|y\|_{\mathcal{B}}$, and define $\hat{x}_i^*(\cdot)$ by:

$$\hat{x}_i^*(y) = \frac{\|x_i\|_{\mathcal{B}}^2}{\|x_i\|_{\mathcal{H}}^2} (y, x_i)_{\mathcal{H}}$$

By the Hahn-Banach Theorem, $\hat{x}_i^*(\cdot)$ has an extension $x_i^*(\cdot)$ to \mathcal{B} , such that $|x_i^*(y)| \leq p_i(y) = \|x_i\|_{\mathcal{B}} \|y\|_{\mathcal{B}}$ for all $y \in \mathcal{B}$. By definition of $p_i(\cdot)$, we see that $\|x_i^*\|_{\mathcal{B}^*} \leq \|x_i\|_{\mathcal{B}}$. On the other hand $x_i^*(x_i) = \|x_i\|_{\mathcal{B}}^2 \leq \|x_i\|_{\mathcal{B}} \|x_i^*\|_{\mathcal{B}^*}$,

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so that $x_i^*(\cdot)$ is a duality mapping for x_i . If $x_i^*(x) = 0$ for all i , then $x \in \bigcap_{i=1}^{\infty} \hat{M}_i = \{0\}$ so that the family $\{x_i^*\}_{i=1}^{\infty}$ is total. If we let $\|x_i\|_{\mathcal{B}} = 1$, it is clear that $x_i^*(x_j) = \delta_{ij}$, for all $i, j \in \mathbb{N}$. Thus, $\{x_i, x_i^*\}$ is an M-basis system with $\|x_i\|_{\mathcal{B}} \|x_i^*\|_{\mathcal{B}^*} = 1$ for all i . \square

CONCLUSION

In this paper we have first shown that every infinite dimensional separable Banach space is very close to a Hilbert space in a well defined manner, providing the best possible improvement on the well-known theorem of Mazur. We have then provided a solution to the M-basis problem by showing that every infinite dimensional separable Banach space has a M-basis $\{x_i, x_i^*\}$, with the property that $\|x_i\|_{\mathcal{B}} \|x_i^*\|_{\mathcal{B}^*} = 1$ for all i .

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